Duality Riesz representation on L^p and Riesz-Markov-Kakutani

I derive Riesz rep on Hilbert spaces in my 2025 Spring MTH 403 notes.
This uses material from Royden & Le Gall.
(section 8.1)

Royden only does it for $L^p(E)$ where $E \subset \mathbb{R}$ is a meas. set.

<u>Theorem</u> Let $(\Omega, \mathcal{F}, \mu)$ be a finite meas. space, $1 \le p < \infty$, $q = p^*$. If T is a bounded linear functional $L^p(\Omega)$, $\exists$ $g \in L^q$ st

$$T(f) = \int f \cdot g \quad , \text{ and } \|T\|_{op} = \|g\|_q$$

Pf: Given T, bounded linear operator.

Define $\nu(A) = T(1_A)$, show ν is a signed measure

that is AC wrt μ, $\Rightarrow$ $\exists$ $g \in L^1$ st $\nu(A) = \int_A g \, d\mu$

So $\forall$ simple φ, $T(\varphi) = \sum a_i T(1_{E_i}) = \sum a_i \int_{E_i} g \, d\mu = \int g \varphi \, d\mu$

Rest is by approximation.

Cor: $(L^p)^* = \{ T : T \text{ is a bounded linear functional on } L^p \} \cong L^q$
(with operator norm) isometric isomorphism

Rem: L^∞ is different. $(L^\infty)^* \cong$ space of bounded, finitely additive signed measures ν, $\nu \ll \mu$, with total variation norm.

Cor: Let $1<p<\infty$. Then $(L^p)^{**} = (L^q)^* = L^p$. In other words $L^p(\Omega, \mathcal{F}, \mu)$ is a reflexive Banach space.

Weak-convergence: Let $\{f_n\}$ be a sequence in L^p. Then $f_n \xrightarrow{w} f \in L^p$ if $T(f_n) \to T(f)$ $\forall\, T \in (L^p)^*$; ie $\int f_n g \to \int f g$ $\forall\, g \in (L^q)^*$

Exer: If $f_n \to f \in L^p$ in norm, then $f_n \xrightarrow{w} f$

Weak-$\star$ convergence: Let X be a Banach space, X^* its dual. If $T \in X^*$,

then $J_x : X^* \to \mathbb{C}$ or $\mathbb{R}$ defined by $J(T) = T(x)$ (evaluation functional) is a bounded linear functional.

Let $\{T_n\} \in X^*$ we say $T_n \xrightarrow{w\text{-}\star} T$ if $J_x(T_n) \to J_x(T)$ $\forall x \in X$

If $X = X^{**}$ (is reflexive) then weak and weak-$\star$ convergence are the same.

If $1<p<\infty$ then $(L^p)^{**} = L^p$, so above applies.

Its the L^∞ case thats interesting since its dual is the space of measures.

We will have use for weak-$\star$ convergence in a slighty different setting, but we will need to introduce a topology on Ω for this!

(Will explore this in Ex 6.4 in Le Gall)

Topological space X.

Topology: $\phi, X \in T$, closed under arbitrary unions and finite intersections

Separable space: $\exists \; S = \{x_i\}_{i=1}^{\infty}$ st every $A \in T$ contains an elmt of S.

Hausdorff space: Distinct points have disjoint neighborhoods (or open sets containing them)

Compact: Every open cover has a finite subcover.

Locally compact: every pt has a compact neighborhood

Borel measure on a topological space: the σ-algebra $\mathcal{F}$ contains T and $\mu(K) < \infty \quad \forall$ compact K.

Radon measure: Borel measures that are both inner and outer regular.

Rem: Topologies are not typically semirings.

If (E, d) is a metric space and T is the metric topology, then Borel measures are Radon measures some conditions

Thm: (21.10 in Royden) If μ is a Radon pre measure on X, T a Hausdorff space, then μ^* the Caratheodory extension is a Radon measure.

Riesz-Markov theorem: let X be locally compact Hausdorff and I a positive linear functional on $C_c(X)$. Then $\exists$ a unique Radon measure $\hat{\mu}$ on $\mathcal{B}(X)$ st

$$I(f) = \int f \, d\hat{\mu} \qquad \forall f \in C_c(X)$$

(Thm 3.13 in Le Gall, 21.4 in Royden)

However unless X itself is compact $C_c(X)$ is NOT a Banach space.

Riesz-Markov Kakutani: Let X be locally compact seperable space, let $C_0(X) = \{f \in C(X) : \forall \epsilon > 0 \; \exists \; K, \text{compact st } |f(x)| < \epsilon, x \in X \setminus K\}$ This is a Banach space with sup norm. Let Φ be a continuous linear functional on $C_0(E)$ then $\exists$ a unique signed measure $\hat{\mu}$ on $(X, \mathcal{B}(X))$

$$\Phi(f) = \int f \, d\hat{\mu}$$

$M(X)$ = space of signed measures is a linear space and

The $|\mu|$ function; ie if $\mu = \mu^+ + \mu^-$, then $|\mu|(X) = \mu^+(X) + \mu^-(X)$ is a norm.

X: So we have $Y = C_0(X)$, $Y^* = \{\mu : \mu \text{ is a signed meas. on } X\}$
So $\mu_n \xrightarrow{w^*} \mu$ if $\int f \, d\mu_n \to \int f \, d\mu \quad \forall f \in Y$.

weak-$*$ convergence is a mode of convergence for measures.

Notes to self
Legall has some proofs of $|\mu|$ being a meas, but these are hard and long.
The Δ inequality is much more obvious from LeGalls definition

$$|\mu + \nu|(X) = \sup_{A_n} \sum |\mu + \nu \; (A_n)|$$

$$\leq \sup_{A_n} \sum |\mu(A_n)| + |\nu(A_n)| \leq |\mu|(X) + |\nu|(X)$$

So not a good exercise at the moment. If you knew the Hahn-Jordan decomp it would be easier.

Convergence in distribution : $X_n \xRightarrow{d} X$ if $F_n(t) \longrightarrow F(t)$ $t \in$ continuity point of F.

Thm : $X_n \xRightarrow{d} X$ iff $\int f(t)\, d\mu_{X_n}(t) \longrightarrow \int f(t)\, d\mu_X(t)$

$\forall f \in C_b(\mathbb{R})$

For more general random variables$_n$ ($X = \mathbb{R}^d$, $C[0,1]$ etc.) F_n is more cumbersome, so we will define weak-convergence directly for their distributions μ_n.
That is,

$$\mu_n \xRightarrow{d} \mu \quad \text{if} \quad \int f\, d\mu_n \longrightarrow \int f\, d\mu \quad \forall f \in C_b(X).$$

Rem: The Riesz Markov Kakutani theorem does not apply to C_b. So this is an issue. Here is the common counterexample:

Let $f \in C_0(\mathbb{R})$, and $\mu_n = \delta_{\{n\}}$ the Delta measure at $\{n\}$.

Then $\int f \, \delta_{\{n\}} \longrightarrow 0 = \int f \, d0$ ← the 0 measure

So $\delta_{\{n\}} \xrightarrow{w^*} 0$ (by Riesz-Markov Kakutani)

But for $f \in C_b(\mathbb{R})$ $\int f \, \delta_{\{n\}}$ does not converge to anything.

So vague convergence is a weaker notion of convergence; things can escape off to ∞.

So typically, if you look at Billingsley for example,

$$\text{conv in distribution} \iff \underbrace{\text{weak}^* \text{ convergence on } C_0(\mathbb{R})}_{\text{Vague convergence}} \; + \text{tightness}$$

The issue is the noncompactness of $\mathbb{R}$. If X is compact then $C_c(X) = C_b(X) = C_0(X)$. In fact, you have to simply define $C_0(X) = C_b(X)$, since there is no notion of "decaying to 0 at ∞".

Q: Is $C_b(\mathbb{R}^d)$ a Banach space? Yes. What is $C_b(\mathbb{R}^d)^*$? It is the space of Radon measures on the Stone-Cech compactification of $\mathbb{R}^d$.

So weak convergence is also a weak-$\star$ convergence.

Note that the Riesz-Markov-Kakutani theorem applies to $C(K)$ with K compact Hausdorff.

Then $C_b(\mathbb{R}) \longleftrightarrow C(\beta\mathbb{R})$

compactification

$C(\beta\mathbb{R})^*$ is described by Riesz Markov Kakutani.

(See Jerison, Rings of Continuous functions)

If K is not compact then $C_b(K)^*$ can still be identified with <u>finitely additive</u> regular signed Borel measures.

To get countably additive, need compactness.

Tightness: Let $(X, \mathcal{T})$ be a Hausdorff space and $\mathcal{F}$ be a σ-field that contains $\mathcal{T}$. Let M be a collection of signed meas. M is called tight if $\forall \varepsilon > 0$, $\exists K_\varepsilon \subset X$ compact st $\forall \mu \in M$,

$$|\mu|(X \setminus K_\varepsilon) < \varepsilon \qquad \text{—☆1}$$

If $M = \{\mu\}$, then this is the same as inner regularity.

Prop: Let $\{\mu_n\}$ be tight probability meas. and $\mu_n \xrightarrow{w\text{-}*} \mu$. Then $\mu_n \overset{d}{\Rightarrow} \mu$

Pf: The idea is easy, you start with $f \in C_b(X)$, choose a compact set K_ε st ☆1 holds and modify f so that it's in $C_0(X)$

BUT to construct these fns, need some general topology:

Urysohn 1) $(X, \mathcal{T})$ (at least on compact subsets) is metrizable.

Urysohn: A topological space is metrizable iff it's normal.

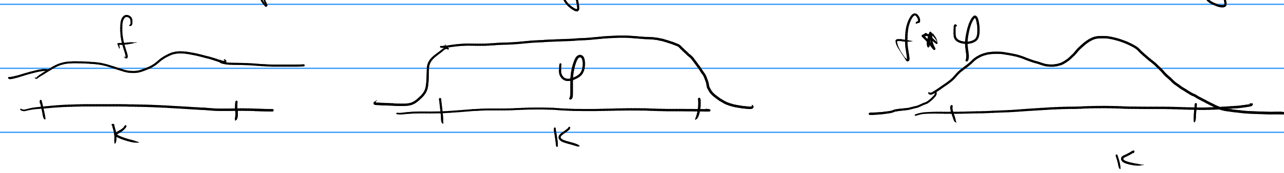
Most of this is in Royden. The upshot is that if $(X, \mathcal{T})$ is compact, it is normal. So when spaces are locally compact, you have a metric to work with. So you might as well work with metric spaces.

In fact, this is what Billingsley does! He works with measures on complete separable metric spaces.

2) Use the metric to construct bump fns: Let $K \subset U$ where K compact, U open. $\exists$ a bump fn ϕ that is 1 on K and vanishes outside of U. U is usually chosen as: $\{x : d(x,K) < \epsilon\}$

3) Convolve f with a bump fn of K to get $f \in C_0$.

This is painful to do in general (need to recall Urysohn etc)



HENCEFORTH WORK on $\mathbb{R}^d$, since all random variables and their distributions are $\mathbb{R}^d$ valued.

Related is also Prokhorov's theorem: $\{\mu_\alpha\}$ is precompact (sequentially compact) in the topology of weak convergence / conv in distribution iff its tight.

<u>Prop</u> / Exercise: Suppose $\mu_n \Rightarrow \mu$, then $\{\mu_n\}$ are tight.

Rem (see Khoshnevisan Thm 7.7): let $\{\mu_n\}$ be Prob measures and μ be a prob. measure. Then $\mu_n \xRightarrow{d} \mu$ iff $\int f\, d\mu_n \to \int f\, d\mu$ $\forall f \in C_c(\mathbb{R})$. This theorem is also in Le Gall. We will do this soon.

Implicit in the proof is that μ is a prob. measure and plays an IMPORTANT ROLE. He shows, by approximation that if

$$\int f\, d\mu_n \longrightarrow \int f\, d\mu \quad \forall f \in C_c(\mathbb{R}) \quad \text{then}$$

$$\int f\, d\mu_n \longrightarrow \int f\, d\mu \quad \forall f \in C_b(\mathbb{R})$$

A similar theorem can be proven for all finite measures.